Cor : $P \in U \hookrightarrow X$: variety $\Rightarrow \exists$ affine V s.t. $P \in V \hookrightarrow U$.

Pf: By replace X with an affine open subvariety containg P we may assume $X \subseteq \mathbb{A}^n$ is affine.

$Z := X \setminus U \hookrightarrow X \quad Z' = Z \cup \{P\} \hookrightarrow X$

$I(Z') \subsetneqq I(Z) \Rightarrow \exists F \in k[x_1, \cdots, x_n]$ s.t.

$f = F \bmod I \in I(Z) \setminus I(Z')$

i.e. $f(P) \neq 0$ & $f(Q) = 0 \ \forall Q \in Z$.

$\Rightarrow P \in X_f \subset U$. $\square$

§ 6.4 Products and Graphs

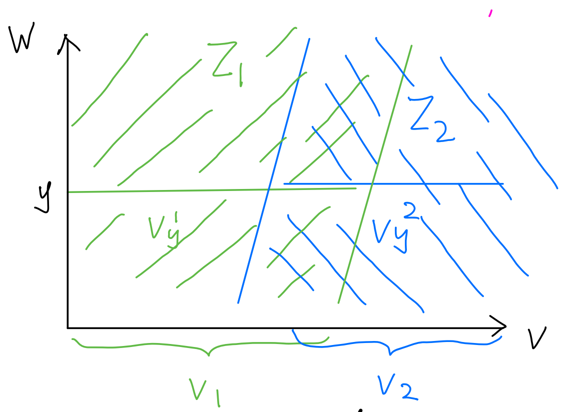
$\mathcal{U}_{i_1} \times \cdots \times \mathcal{U}_{i_r} \times \mathbb{A}^n \hookrightarrow A = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^n$ affine covering

$\mathcal{U}_{j_1} \times \cdots \times \mathcal{U}_{j_s} \times \mathbb{A}^m \hookrightarrow B = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_r} \times \mathbb{A}^m$ affine covering

$\Rightarrow \mathcal{U}_{i_1} \times \cdots \times \mathcal{U}_{i_r} \times \mathcal{U}_{j_1} \times \cdots \times \mathcal{U}_{j_s} \times \mathbb{A}^{m+n} \hookrightarrow A \times B$ affine covering.

Prop 6. $V \hookrightarrow A$, $W \hookrightarrow B$ subvarieties $\Rightarrow$ $V \times W \hookrightarrow A \times B$ subvariety.

Pf: difficulty: $V \times W$ is irreducible.



1° $V_i := \{ x \mid x \times W \subset Z_i \}$

2° $V = V_1 \cup V_2$ (Since W is irr)

3° $V_1 = \bigcap_y V_y^1$ closed

$V_2 = \bigcap_y V_y^2$ closed

4° $V = V_1 \cup V_2 \Rightarrow V = V_i \Rightarrow V \times W \subset Z_i$ □

Prop 7: X, Y as above

1) $pr_1 : X \times Y \to X$ & $pr_2 : X \times Y \to Y$ are morphisms

2). $f: Z \to X$, $g: Z \to Y$ morphisms $\Rightarrow$ $(f,g): Z \to X \times Y$ morphism

$z \mapsto (f(z), g(z))$

3). $f: X' \to X$, $g: Y' \to Y$ morphisms $\Rightarrow$ $f \times g : X' \times Y' \to X \times Y$ morphism

$(x', y') \mapsto (f(x'), g(y'))$

4). $\Delta_X := \{ (x,y) \in X \times X \mid x = y \} \hookrightarrow X \times X$

$\delta_X : X \to \Delta_X$ is an isomorphism.

$x \mapsto (x, x)$

⑩ Pf: reduce to affine case. left to the reader

Cor: $f, g : X \to Y$ morphism of varieties. Then

$$Z = \{ x \in X \mid f(x) = g(x) \} \hookrightarrow X$$

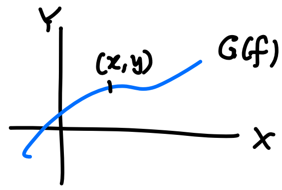
if Z dense in X, then $f = g$.

Pf:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, hook] & \Delta_Y \arrow[d, hook] \\
X \arrow[r, "{(f,g)}"] & Y \times Y
\end{tikzcd}$$

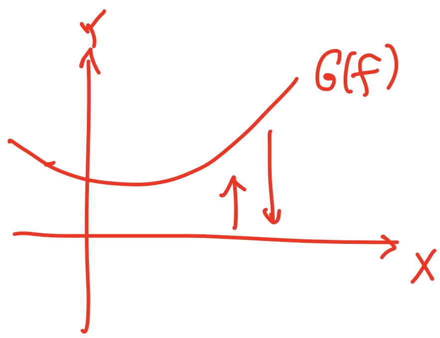
Def: $f : X \to Y$ morphism of varieties. the graph of f is defined ~~to~~ be

$$G(f) = \{ (x, y) \in X \times Y \mid y = f(x) \}$$



Prop 8: (1) $G(f) \overset{\text{Var}}{\hookrightarrow} X \times Y$.

(2)
$$\begin{tikzcd}
G(f) \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & X \times Y \arrow[r] & X
\end{tikzcd}$$



Pf:

$$\begin{tikzcd}
 & G(f) \arrow[r] \arrow[d, hook] & \Delta_Y \arrow[d, hook] \\
X \arrow[ur, dotted] \arrow[r, "{(id, f)}"] \arrow[dr, "id"'] & X \times Y \arrow[r, "{(f, id)}"] \arrow[d, "pr_1"] & Y \times Y \\
 & X &
\end{tikzcd}$$

Algebraic function fields and dimension of varieties

K/k = f.g. field ext. $(k = \bar{k})$

$\mathrm{tr.deg}_k K$ = transcendence degree of K over k

:= the smallest n s.t. $\exists\, x_1, \cdots, x_n \in K$ s.t. $K/k(x_1, \cdots, x_n)$ = algebraic

In this case, we call K an algebraic function field in n variables $/k$

Def: X = var. $k(X)$ = f.g. ext. of k. dimension of X is defined as

$$\dim(X) := \mathrm{tr.deg}_k\, k(X).$$

curve = var. of dim 1 (plane curve could be not irr & not red.)

surface = var. of dim 2

$\vdots$

This definition is consistent with intuition.

Prop 10. X = variety

(1). $\phi \neq U \overset{\text{open}}{\hookrightarrow} X. \Rightarrow \dim U = \dim X$

(2). X = affine $\Rightarrow \dim X = \dim X^*$ (X^* = projective closure)

(3). $\dim X = 0 \Leftrightarrow X = pt.$

(4). $X \underset{\neq}{\hookrightarrow}$ curve $\Rightarrow X = pt$

(5). $X \hookrightarrow \mathbb{A}^2$ (or $\mathbb{P}^2$) $\Rightarrow \dim X = 1$ iff X = plane curve.

equivalent definitions of dimension <u>commutative algebra</u>